Exercise 1

Calculate the length of $\vec{u} = (1, 0, 1)$, the distance between $\vec{u}$ and $\vec{v} = (0, 1, -1)$ and the angle between $\vec{u}$ and $\vec{v}$

(i) relative to the standard dot product

standard dot product ; $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_N v_N$

$$= \sum_{i=1}^{n} u_i v_i$$

→ length of $\vec{u}$ = norm

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

$u_1 = 1$

$u_2 = 0$ $\Rightarrow \|\vec{u}\| = \sqrt{(1)^2 + (0)^2 + (1)^2} = \boxed{\sqrt{2}}$

$u_3 = 1$

→ distance between $\vec{u}$ and $\vec{v}$

$$\|\vec{u} - \vec{v}\| = \sqrt{\langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle} = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$$

$$\|\vec{u} - \vec{v}\| = \sqrt{(1-0)^2 + (0-1)^2 + (1-(-1))^2} = \sqrt{1 + 1 + 4}$$

$$\boxed{\|\vec{u} - \vec{v}\| = \sqrt{6}}$$

→ angle between $\vec{u}$ and $\vec{v}$

$$\angle(\vec{u}, \vec{v}) = \cos^{-1} \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|} = \cos^{-1} \frac{\vec{u} \cdot \vec{v}}{\sqrt{\vec{u} \cdot \vec{u}} \sqrt{\vec{v} \cdot \vec{v}}}$$

$$= \cos^{-1} \left[\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{\sqrt{2} \sqrt{(0)^2 + (1)^2 + (-1)^2}} \right]$$

$$= \cos^{-1} \left[\frac{(1)(0) + (0)(1) + (1)(-1)}{\sqrt{2} \sqrt{2}} \right] = \cos^{-1} \left[\frac{-1}{2} \right]$$

$$= \boxed{120}^\circ$$

(ii) Relative to the inner product given by $\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + 2u_2 v_2 + 2u_3 v_3$

$\vec{u} = (1, 0, 1)$

$\rightarrow \|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle} = \sqrt{u_1^2 + 2u_2^2 + 2u_3^2}$

$\|\vec{u}\| = \sqrt{(1)^2 + 2(0)^2 + 2(1)^2} = \sqrt{1 + 2} = \boxed{\sqrt{3}}$

$\rightarrow \|\vec{u} - \vec{v}\| = \sqrt{\langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle}$

$= \sqrt{(u_1 - v_1)^2 + 2(u_2 - v_2)^2 + 2(u_3 - v_3)^2}$

$= \sqrt{(1 - 0)^2 + 2(0 - 1)^2 + 2(1 - (-1))^2}$

$= \sqrt{1 + 2 + 8} = \boxed{\sqrt{11}}$

$\rightarrow \angle(\vec{u}, \vec{v}) = \cos^{-1} \dfrac{\langle \vec{u}, \vec{v} \rangle}{\sqrt{\langle \vec{u}, \vec{u} \rangle} \sqrt{\langle \vec{v}, \vec{v} \rangle}} = \cos^{-1} \dfrac{(u_1 v_1 + 2u_2 v_2 + 2u_3 v_3)}{\sqrt{\langle \vec{u}, \vec{u} \rangle} \sqrt{\langle \vec{u}, \vec{v} \rangle}}$

$\langle \vec{u}, \vec{u} \rangle = u_1^2 + 2u_2^2 + 2u_3^2 = (1)^2 + 2(0)^2 + 2(1)^2 = 3$

$\langle \vec{v}, \vec{v} \rangle = v_1^2 + 2v_2^2 + 2v_3^2 = (0)^2 + 2(1)^2 + 2(-1)^2 = 4$

$\Rightarrow \angle(\vec{u}, \vec{v}) = \cos^{-1} \left(\dfrac{(1)(0) + 2(0)(1) + 2(1)(-1)}{\sqrt{3} \sqrt{4}} \right)$

$\theta = \cos^{-1} \left(\dfrac{-2}{\sqrt{12}} \right) = \boxed{125.26^\circ}$

Exercise 2

Which of the following bases are orthogonal and which are orthonormal relative to the dot product?

(i) $(-1,1)$ $(0,2)$

→ two vectors $\vec{u}, \vec{v}$ are orthogonal relative to an inner product $\langle \vec{u}, \vec{v} \rangle$ if $\langle \vec{u}, \vec{v} \rangle = 0$

$$\vec{u} = (-1,1)$$
$$\vec{v} = (0,2)$$

$\vec{u} \cdot \vec{v} = (-1)(0) + (1)(2) = 2 \Rightarrow$ not orthogonal

(ii) $(0,0,1)$ $(2,-2,0)$ $(1,1,0)$

$$\underset{\vec{u}}{\downarrow} \qquad \underset{\vec{v}}{\downarrow} \qquad \underset{\vec{w}}{\downarrow}$$

→ need to take dot product between each pair

$$\vec{u} \cdot \vec{v} = (0)(2) + (0)(-2) + (1)(0)$$
$$= 0 + 0 + 0$$
$$= 0$$

$$\vec{u} \cdot \vec{w} = (0)(1) + (0)(1) + (1)(0)$$
$$= 0 + 0 + 0$$
$$= 0$$

$$\vec{v} \cdot \vec{w} = (2)(1) + (-2)(1) + (0)(0)$$
$$= 2 - 2 + 0$$
$$= 0$$

→ they are orthogonal.

Now we check to see if they are orthonormal.

$\vec{u} \cdot \vec{u} = (0,0,1) \cdot (0,0,1)$

$= (0)(0) + (0)(0) + (1)(1)$

$= 1$

$\vec{v} \cdot \vec{v} = (2)(2) + (-2)(-2) + (0,0)$

$= 4 + 4 + 0 = 8$

$\vec{w} \cdot \vec{w} = (1)(1) + (1)(1) + (0)(0)$

$= 2$

$\Rightarrow$ they are not orthonormal.
but they are orthogonal

(iii) $(1,0,0)$ $(0, 3/5, 4/5)$ $(0, 4/5, -3/5)$

$\downarrow$ $\vec{u}$ $\downarrow$ $\vec{v}$ $\downarrow$ $\vec{w}$

$\vec{u} \cdot \vec{v} = (1,0,0) \cdot (0, 3/5, 4/5)$

$= (1)(0) + (0)(3/5) + (0)(4/5)$

$= 0 + 0 + 0$

$= 0$

$\vec{u} \cdot \vec{w} = (1, 0, 0) \cdot (0, 4/5, -3/5)$

$= (1)(0) + (0)(4/5) + (0)(-3/5)$

$= 0$

$\vec{v} \cdot \vec{w} = (0, 3/5, 4/5) \cdot (0, 4/5, -3/5)$

$= (0)(0) + (3/5)(4/5) + (4/5)(-3/5)$

$= 0 + 12/25 - 12/25$

$= 0$ $\Rightarrow$ they are orthogonal

$$\vec{u} \cdot \vec{u} = (1)(1) + (0)(0) + (0)(0)$$
$$= 1$$

$$\vec{v} \cdot \vec{v} = (0)(0) + (3/5)(3/5) + (4/5)(4/5)$$
$$= 0 + 9/25 + 16/25$$
$$= 1$$

$$\vec{w} \cdot \vec{w} = (0)(0) + (4/5)(4/5) + (-3/5)(-3/5)$$
$$= 0 + 16/25 + 9/25$$
$$= 1$$

$\Rightarrow$ they are orthonormal relative to the dot product.

Exercise 3

Calculate the coordinates of $\vec{v} = (3, -2, -1)$ relative to the orthogonal basis

$$\{ (-2, 0, 0), (0, 2, 3), (0, 3, -2) \} :$$

(i) relative to the standard dot product

$$\vec{v} = k_1 \vec{v_1} + k_2 \vec{v_2} + k_3 \vec{v_3}$$

where $\vec{v_1} = (-2, 0, 0)$
$\vec{v_2} = (0, 2, 3)$
$\vec{v_3} = (0, 3, -2)$

if the basis B is orthogonal we can write;

$$k_j = \frac{\langle \vec{v}, \vec{v_j} \rangle}{\langle \vec{v_j}, \vec{v_j} \rangle} = \frac{\langle \vec{v}, \vec{v_j} \rangle}{\| \vec{v_j} \|^2}$$

where k_j are the coordinates of the vector $\vec{v}$ relative to the orthogonal basis B

$$k_1 = \frac{\langle \vec{V}, \vec{v}_1 \rangle}{\| \vec{v}_1 \|^2} \qquad \vec{v} = (3,-2,-1) \qquad \vec{v_2} = (0,2,3)$$
$$\vec{v}_1 = (-2,0,0) \qquad \vec{v_3} = (0,3,-2)$$

$$k_1 = \frac{(3)(-2) + (-2)(0) + (-1)(0)}{\left(\sqrt{(-2)^2 + (0)^2 + (0)^2}\right)^2} = \frac{-6}{4}$$

$$k_2 = \frac{(3)(0) + (-2)(2) + (-1)(3)}{\left(\sqrt{(0)^2 + (2)^2 + (3)^2}\right)^2} = \frac{-4-3}{4+9} = \frac{-7}{13}$$

$$k_3 = \frac{(3)(0) + (-2)(3) + (-1)(-2)}{\left(\sqrt{(0)^2 + (3)^2 + (-2)^2}\right)^2} = \frac{-6+2}{9+4} = \frac{-4}{13}$$

$\Rightarrow$ coordinates relative to the standard dot product:

$$\boxed{\left(-\tfrac{6}{4}, -\tfrac{7}{13}, -\tfrac{4}{13}\right)}$$

(ii) relative to the inner product $\langle \vec{u}, \vec{v} \rangle = 4u_1v_1 + u_2v_2 + 4u_3v_3$

$\longrightarrow$ orthogonal relative to this inner product?

$$\vec{v_1} = (-2,0,0)$$
$$\vec{v_2} = (0,2,3)$$
$$\vec{v_3} = (0,3,-2)$$

$$\vec{v_1} \cdot \vec{v_2} = (-2)(0) + (0)(2) + (0)(3) = 0$$
$$\vec{v_1} \cdot \vec{v_3} = (-2)(0) + (0)(3) + (0)(-2) = 0$$
$$\vec{v_2} \cdot \vec{v_3} = (0)(0) + (2)(3) + (3)(-2) = 0$$

basis is $\Rightarrow$ still orthogonal relative to this inner product.

$$k_1 = \frac{\langle \vec{v}, \vec{v_1} \rangle}{\| \vec{v_1} \|^2}$$

$$\vec{v} = (3, -2, -1)$$
$$\vec{v_1} = (-2, 0, 0)$$
$$\vec{v_2} = (0, 2, 3)$$
$$\vec{v_3} = (0, 3, -2)$$

$$\boxed{\langle \vec{u}, \vec{v} \rangle = 4 u_1 v_1 + u_2 v_2 + u_3 v_3}$$

$$k_1 = \frac{4(3)(-2) + (-2)(0) + (-1)(0)}{\left(\sqrt{\langle \vec{v_1}, \vec{v_1} \rangle} \right)^2}$$

$$k_1 = \frac{-24}{\left(\sqrt{4(-2)(-2) + (0)(0) + (0)(0)} \right)^2} = -\frac{24}{16}$$

$$k_2 = \frac{4(3)(0) + (-2)(2) + (-1)(3)}{\left(\sqrt{4(0)(0) + (2)(2) + (3)(3)} \right)^2}$$

$$k_2 = \frac{-4 - 3}{4 + 9} = -\frac{7}{13}$$

$$k_3 = \frac{4(3)(0) + (-2)(3) + (-1)(-2)}{\left(\sqrt{4(0)(0) + (3)(3) + (-2)(-2)} \right)^2}$$

$$k_3 = \frac{-6 + 2}{9 + 4} = -\frac{4}{13}$$

$\Rightarrow$ coordinates relative to this inner product are

$$\boxed{\left(-24/16, \ -7/13, \ -4/13 \right)}$$

Exercise 4

Find the orthogonal projection of the vector $\vec{v}$ onto the plane spanned by the orthogonal basis $\{\vec{u_1}, \vec{u_2}\}$ (relative to the dot product), where

$\vec{u_1} = (-1, 2, 0)$, $\vec{u_2} = (2, 1, -1)$

and

(i) $\vec{v} = (1, 0, 1)$

$$\text{proj}_w \vec{v} = \frac{\langle \vec{v}, \vec{u_1} \rangle}{\|\vec{u_1}\|^2} \vec{u_1} + \frac{\langle \vec{v}, \vec{u_2} \rangle}{\|\vec{u_2}\|^2} \vec{u_2}$$

$$\frac{\langle \vec{v}, \vec{u_1} \rangle}{\|\vec{u_1}\|^2} = \frac{(1)(-1) + (0)(2) + (1)(0)}{\left(\sqrt{(-1)^2 + (2)^2 + (0)^2}\right)^2} = \frac{-1}{5}$$

$$\frac{\langle \vec{v}, \vec{u_2} \rangle}{\|\vec{u_2}\|^2} = \frac{(1)(2) + (0)(1) + (1)(-1)}{\left(\sqrt{(2)^2 + (1)^2 + (-1)^2}\right)^2} = \frac{2 - 1}{4 + 1 + 1} = \frac{1}{6}$$

$$\Rightarrow \text{proj}_w \vec{v} = -\frac{1}{5}(-1, 2, 0) + \frac{1}{6}(2, 1, -1)$$

$$\text{proj}_w \vec{v} = \left(\tfrac{1}{5}, -\tfrac{2}{5}, 0\right) + \left(\tfrac{2}{6}, \tfrac{1}{6}, -\tfrac{1}{6}\right)$$

$$= \boxed{\left(\tfrac{8}{15}, -\tfrac{7}{30}, -\tfrac{1}{6}\right)}$$

(ii) $\vec{v} = (1, -1, 1)$

$$\text{proj}_W \vec{v} = \frac{\langle \vec{v}, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \vec{u}_1 + \frac{\langle \vec{v}, \vec{u}_2 \rangle}{\|\vec{u}_2\|^2} \vec{u}_2$$

$$\frac{\langle \vec{v}, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} = \frac{(1)(-1) + (-1)(2) + (1)(0)}{\left(\sqrt{(-1)^2 + (2)^2 + (0)^2}\right)^2} = \frac{-1 -2}{1 + 4} = -3/5$$

$$\frac{\langle \vec{v}, \vec{u}_2 \rangle}{\|\vec{u}_2\|^2} = \frac{(1)(2) + (-1)(1) + (1)(-1)}{\left(\sqrt{(2)^2 + (1)^2 + (-1)^2}\right)^2} = \frac{2 - 1 - 1}{4 + 1 + 1} = 0$$

$$\text{proj}_W \vec{v} = -3/5 \; (-1, 2, 0) + 0 \; (2, 1, -1)$$

$$= \boxed{(3/5, -6/5, 0)}$$